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Properties of Some Polynomial Projections

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1. INTRODUCTION

Let $\mathscr{P}_n$ denote the class of all *projections*, i.e., operators which are bounded and idempotent, mapping the space $C[-1, 1]$ onto the subspace Π_n of polynomials of degree $\leqslant n$.

The quality of the approximations obtained from a projection $P \in \mathscr{P}_n$ is governed by the inequality

$$\|f - Pf\|_\infty \leqslant (1 + \|P\|)\, E_n(f),$$

where $E_n(f)$ is the error of the best approximation of f by elements of Π_n.

It is known [4] that there exists $P^* \in \mathscr{P}_n$ such that

$$\|P^*\| \leqslant \|P\|$$

for all $P \in \mathscr{P}_n$. Such a P^* is called a *minimal projection* from the class $\mathscr{P}_n$. Discovering such a projection is, however, very difficult. The complete solution to this problem, even in the case $n = 2$, remains unknown.

The *Fourier–Chebyshev operator* $S_n \in \mathscr{P}_n$ is defined by

$$S_n f = \sum_{k=0}^{n}{}' a_k[f]\, T_k \qquad (f \in C[-1, 1]), \tag{1.1}$$

where

$$a_k[f] = \frac{2}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} f(x)\, T_k(x)\, dx \qquad (k = 0, 1,\ldots); \tag{1.2}$$

$$T_k(x) = \cos(k \arccos x). \tag{1.3}$$

The symbol $\sum'$ denotes the sum with the first term halved.

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For every operator $P \in \mathscr{P}_n$ we have

$$\| P \| \geqslant \sigma_n , \tag{1.4}$$

where

$$\sigma_n = \tfrac{1}{2} \| S_0 + S_n \| \tag{1.5}$$

(see [2]). Note that $\frac{1}{2}(S_0 + S_n) \notin \mathscr{P}_n$.

Paszkowski [5, p. 84] gave the exact expression

$$\sigma_n = \frac{[n/2] + 1}{2[n/2] + 1} + \frac{1}{\pi} \left(n \sum_{k=1}^{[n/2]} \frac{1}{(2k-1)(n-2k+1)} \cot \frac{(2k-1)\pi}{2n} \right.$$
$$\left. - \sum_{k=1}^{[(n+1)/2]} \frac{n - 4k + 3}{(2k-1)(n-2k+2)} \cot \frac{(2k-1)\pi}{2n+2} \right) \tag{1.6}$$

and the asymptotic form

$$\sigma_n = \frac{2}{\pi^2} \log n + O(1). \tag{1.7}$$

In this paper we investigate the subclass $\mathscr{T}_{np}$ ($p \geqslant 0$) of the class $\mathscr{P}_n$, defined as the set of projections of the form

$$Pf = S_n f + \sum_{l=1}^{p} a_{n+l}[f]\, q_l \qquad (f \in C[-1, 1]), \tag{1.8}$$

where $q_1, q_2, \ldots, q_p$ can be arbitrary elements of Π_n [3].

Phillips *et al.* [6] gave a theorem characterizing minimal projections from the class $\mathscr{T}_{np}$ (see Section 2).

In the present paper we give some conditions which are sufficient in order that an operator $P \in \mathscr{T}_{np}$ be a minimal projection from the class $\mathscr{T}_{np}$ for some specified values of n and p. These results (see Section 4) were obtained by application of the theorem of Phillips *et al.* mentioned above.

As a by-product we have obtained a lower bound for the norm of an arbitrary projection $P \in \mathscr{T}_{np}$, which is better than that from (1.4).

2. Results of Phillips *et al.*

The *Lebesgue function* of the operator $P \in \mathscr{T}_{np}$ given by (1.8) is the function

$$\Lambda_P(x) = \frac{2}{\pi} \int_{-1}^{1} | F_P(x, y)| \,(1 - y^2)^{-1/2}\, dy \qquad (-1 \leqslant x \leqslant 1), \tag{2.1}$$

where $F_P(x, y)$ is defined as

$$F_P(x, y) = \sum_{k=0}^{n}{}' T_k(x)\, T_k(y) + \sum_{l=1}^{p} q_l(x)\, T_{n+l}(y) \qquad (-1 \leqslant x, y \leqslant 1). \quad (2.2)$$

It is known that

$$\| P \| = \| \Lambda_P \|_\infty . \quad (2.3)$$

The *critical set* of Λ_P is the set

$$\operatorname{crit}(\Lambda_P) = \{x \in [-1, 1] \mid \Lambda_P(x) = \| \Lambda_P \|_\infty\}. \quad (2.4)$$

Phillips *et al.* [6] proved the following.

THEOREM 2.1. *In order that $P \in \mathscr{T}_{np}$ be a minimal projection from the class $\mathscr{T}_{np}$ it is necessary and sufficient that*

$$\inf_{x \in \operatorname{crit}(\Lambda_P)} \sum_{l=1}^{p} w_l(x)\, h_l(x) \leqslant 0 \quad (2.5)$$

for all choices of $w_1, w_2, \ldots, w_p \in \Pi_n$. Here

$$h_l(x) = \int_{-1}^{1} T_{n+l}(y)(1 - y^2)^{-1/2} \operatorname{sgn} F_P(x, y)\, dy \qquad (l = 1, 2, \ldots, p). \quad (2.6)$$

It is also known that among the minimal projections from $\mathscr{T}_{np}$ there is a *symmetric* projection P such that for $f \in C[-1, 1]$ and $x \in [-1, 1]$ the equation

$$(Pf)(x) = (Pg)(-x)$$

holds, where $g(t) = f(-t)$ for $t \in [-1, 1]$. In other words, we have

$$\inf_{P \in \mathscr{T}_{np}} \| P \| = \inf_{Q \in \hat{\mathscr{T}}_{np}} \| Q \|,$$

where $\hat{\mathscr{T}}_{np}$ denotes the class of all symmetric projections from $\mathscr{T}_{np}$.

It can be seen that $\hat{\mathscr{T}}_{np}$ consists of operators defined by formula (1.8) in which $q_1, q_2, \ldots, q_n \in \Pi_n$ are such that

$$q_l(-x) = (-1)^{n+l} q_l(x) \qquad (l = 1, 2, \ldots, p;\ -1 \leqslant x \leqslant 1) \quad (2.7)$$

(see [3]).

The following theorem results from applying the main theorem from [6].

THEOREM 2.2. *In order that $P \in \hat{\mathcal{T}}_{np}$ be a minimal projection from the class $\hat{\mathcal{T}}_{np}$ it is necessary and sufficient that inequality* (2.5) *holds for all choices of $w_1, w_2, \ldots, w_p \in \Pi_n$ such that*

$$w_l(-x) = (-1)^{n+l} w_l(x) \qquad (l = 1, 2, \ldots, p;\ -1 \leqslant x \leqslant 1). \tag{2.8}$$

3. LEMMAS

Let the function $D_n^{(r)}$ $(n, r = 0, 1, \ldots)$ be defined by the formula

$$D_n^{(r)}(u) = \sum_{k=0}^{n+2^r-1}{}' \,(1 - 2^{-r}(k - n)_+) \cos ku, \tag{3.1}$$

where

$$\begin{aligned} a_+ &= a \qquad (a > 0), \\ &= 0 \qquad (a \leqslant 0). \end{aligned} \tag{3.2}$$

For $r = 0$ formula (3.1) defines the well-known Dirichlet kernel

$$D_n^{(0)}(u) = \sum_{k=0}^{n}{}' \cos ku \qquad (n = 1, 0, \ldots). \tag{3.3}$$

Five lemmas, which we give in this section, state some important properties of $D_n^{(r)}$.

LEMMA 3.1. *For $n, r = 0, 1, \ldots$ we have*

$$D_n^{(r)}(u) = \frac{\sin 2^{r-1}u \sin(n + 2^{r-1})\, u}{2^r(1 - \cos u)} \qquad (u \neq 0, \pm 2\pi, \pm 4\pi, \ldots). \tag{3.4}$$

Proof. First observe that formula (3.1) can be transformed to the form

$$D_n^{(r)}(u) = 2^{-r} \sum_{l=0}^{2^r-1} \sum_{k=0}^{n+l}{}' \cos ku.$$

Hence, in view of the identities

$$\begin{aligned} \sum_{i=0}^{m}{}' \cos iu &= \frac{\sin(m + \frac{1}{2})\, u}{2 \sin(u/2)} \\ \sum_{j=1}^{m} \sin(j - \tfrac{1}{2})\, u &= \frac{1 - \cos mu}{2 \sin(u/2)} \end{aligned} \qquad (u \neq 0, \pm 2\pi, \pm 4\pi, \ldots),$$

formula (3.4) follows. ∎

Let us denote

$$\rho_n^{(r)} = \frac{2}{\pi} \int_0^{\pi} | D_n^{(r)}(u)| \, du \qquad (n, r = 0, 1,...). \tag{3.5}$$

Using (3.4) one can easily obtain the equation

$$\rho_{2m}^{(r)} = \rho_m^{(r-1)} \qquad (m, r = 1, 2,...), \tag{3.6}$$

which implies

$$\rho_{2^r m}^{(r)} = \rho_m^{(0)} \qquad (m, r = 1, 2,...). \tag{3.7}$$

Obviously, $\rho_n^{(0)}$ is the Lebesgue constant (norm) of the operator S_n defined by (1.1). As it is known, the formula

$$\rho_n^{(0)} = \frac{1}{2n+1} + \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k} \tan \frac{k}{2n+1} \qquad (n = 0, 1,...) \tag{3.8}$$

holds (see, e.g., [5, p. 75]).
Before we give a formula for $\rho_n^{(r)}$ $(r < 1)$, analogous to (3.8), observe that if we represent n in the form

$$n = 2^w(2l + 1) \qquad (l, w = 0, 1,...) \tag{3.9}$$

then in view of (3.6) we have

$$\begin{aligned} \rho_n^{(r)} &= \rho_{2l+1}^{(r-w)} && (0 \leqslant w \leqslant r), \\ &= \rho_{2^{w-r}(2l+1)}^{(0)} && (w > r). \end{aligned} \tag{3.10}$$

Thus we have to consider the case of n odd only. For $r = 1$ Geddes and Mason [1] gave the formula

$$\rho_{2l+1}^{(1)} = \frac{4}{\pi} \sum_{k=0}^{l} \frac{1}{2k+1} \tan \frac{(2k+1)\pi}{4l+4} \qquad (l = 0, 1,...). \tag{3.11}$$

We prove the following.

LEMMA 3.2. *For $r = 2, 3,...$ and $l = 0, 1,...$ we have*

$$\begin{aligned} \rho_{2l+1}^{(r)} = {} & 2^{r-2}(1 - s) \left(\frac{2^{r-1}q + 1}{N} - 1 \right) \\ & + \frac{4s}{\pi} \sum_{k=0}^{l+2^{r-1}-1} \frac{1 - 2^{1-r}(k - l)_+}{2k+1} \tan^s \frac{(2k+1)\pi}{2^r} \\ & + \frac{2s}{\pi} \sum_{k=1}^{2l+2^r} \frac{1 - 2^{-r}(k - n)_+}{k} \epsilon_k , \end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\epsilon_k &= 0 && (k = N/d,\ 3N/d, \ldots, N),\\
&= -2^r \tan \frac{k}{2N} && (k = 2N/d,\ 4N/d, \ldots, (d-1)\,N/d),\\
&= \tan^s \frac{kq\pi}{2N} \left(\cos \frac{k(2^r q + 1)\,\pi}{2N} \Big/ \cos \frac{k\pi}{2N} - 1\right)\\
& && (k = 1, 2, \ldots, 2l + 2^r;\ k \neq N/d,\ 2N/d, \ldots, N),
\end{aligned}$$

$$s = (-1)^q, \quad q = [N/2^{r-1}], \quad N = 2l + 2^{r-1} + 1, \tag{3.13}$$

and d is the greatest common divisor of the numbers N and q.

Proof. We have to calculate the integral appearing in (3.5). For $n = 2l+1$ ($l = 0, 1, \ldots$) the function $D_n^{(r)}$ is positive at 0, and, in view of (3.4), changes the sign in the interval $(0, \pi)$ only at the points $i\pi/2^{r-1}$ ($i = 1, 2, \ldots, 2^{r-1} - 1$), $j\pi/N$ ($j = 1, 2, \ldots, N-1$). Observe that

$$\frac{iq\pi}{N} < \frac{i\pi}{2^{r-1}} < \frac{(iq+1)\,\pi}{N} \qquad (i = 1, 2, \ldots, 2^{r-1} - 1),$$

where

$$q = [N/2^{r-1}],$$

the symbol $[x]$ denoting the integer part of x.

Hence

$$\begin{aligned}
\rho_n^{(r)} &= \frac{2}{\pi} \sum_{i=1}^{2^{r-1}} (-1)^{(i-1)(q+1)} \left\{ \int_{(i-1)\pi/2^{r-1}}^{((i-1)q+1)\pi/N} D_n^{(r)}(u)\, du \right.\\
&\quad \left. + \sum_{j=1}^{q-1} (-1)^j \int_{((i-1)q+j)\pi/N}^{((i-1)q+j+1)\pi/N} D_n^{(r)}(u)\, du + (-1)^q \int_{iq\pi/N}^{i\pi/2^{r-1}} D_n^{(r)}(u)\, du \right\}\\
&= (-1)^q \frac{2}{\pi} \sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} \left\{ I\left(\frac{(i-1)\,\pi}{2^{r-1}}\right) \right.\\
&\quad \left. + 2 \sum_{j=1}^{q} (-1)^j I\left(\frac{(i-1)\,q + j}{N}\,\pi\right) + (-1)^{q+1} I\left(\frac{i\pi}{2^{r-1}}\right) \right\},
\end{aligned}$$

where

$$I(u) = \int_0^u D_n^{(r)}(v)\, dv.$$

As, in accordance with (3.1), we have

$$I(u) = \frac{1}{2} u + \sum_{k=1}^{n+2^r-1} \frac{1 - 2^{-r}(k-n)_+}{k} \sin ku,$$

we obtain the formula

$$\rho_n^{(r)} = \frac{(-1)^q}{\pi} \left(\omega + 2 \sum_{k=1}^{n+2^r-1} \frac{1 - 2^{-r}(k-n)_+}{k} (\alpha_k + 2\beta_k) \right), \tag{3.14}$$

where ω, α_k, and β_k have the following meanings:

$$\omega = \sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} \left\{ \frac{(i-1)\pi}{2^{r-1}} + 2 \sum_{j=1}^{q} (-1)^j \frac{(i-1)q+j}{N} \pi - (-1)^q \frac{i\pi}{2^{r-1}} \right\}, \tag{3.15}$$

$$\alpha_k = \sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} \left(\sin \frac{(i-1)\pi}{2^{r-1}} - (-1)^q \sin \frac{i\pi}{2^{r-1}} \right), \tag{3.16}$$

$$\beta_k = \sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} \sum_{j=1}^{q} (-1)^j \sin \frac{((i-1)q+j)k\pi}{N}. \tag{3.17}$$

In the remaining part of the proof we transform the expressions occurring on the right-hand sides of (3.15)–(3.17).

Observe that the right-hand side of (3.15) may be rewritten in the form

$$\pi \left\{ (1 - (-1)^q)(2^{1-r} - q/N) \sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} i + \left(\frac{2}{N} \sum_{j=1}^{q} (-1)^j j - 2^{1-r} + \frac{q}{N} (1 - (-1)^q) \right) \sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} \right\}.$$

Hence,

$$\begin{aligned} \omega &= 0 && (q \text{ even}), \\ &= 2^{r-1}\pi \left(1 - \frac{2^{r-1}q+1}{N}\right) && (q \text{ odd}). \end{aligned} \tag{3.18}$$

It follows from definition (3.16) that

$$\alpha_k = 0 \qquad (k = 2^{r-1}m;\ m = 1, 2, \ldots).$$

As we have the identity

$$\sum_{i=1}^{m} t^i \sin iu = \frac{t \sin u - t^{m+1} \sin(m+1)\, u + t^{m+2} \sin mu}{1 - 2t \cos u + t^2}, \tag{3.19}$$

it follows that

$$\begin{aligned}\alpha_k &= 2(-1)^{q+1} \sum_{i=1}^{2^{r-1}-1} (-1)^{i(q+1)} \sin \frac{ik\pi}{2^{r-1}} \\ &= 2(-1)^{q+1} \frac{(-1)^{q+1} \sin(k\pi/2^{r-1}) + (-1)^{q+1} \sin((2^{r-1}-1)(k\pi/2^{r-1}))}{2(1 + (-1)^q \cos(k\pi/2^{r-1}))} \\ &= (1 - (-1)^k) \frac{\sin(k\pi/2^{r-1})}{1 + (-1)^q \cos(k\pi/2^{r-1})} \\ &\qquad (k = 1, 2,..., n + 2^r - 1;\ k \neq 2^{r-1}, 2^r,...).\end{aligned}$$

Finally we get

$$\left.\begin{aligned}\alpha_{2k} &= 0 \\ \alpha_{2k-1} &= 2 \tan^s \frac{2k-1}{2^r}\, \pi\end{aligned}\right\} \quad (k = 1, 2,..., l + 2^{r-1}), \tag{3.20}$$

where

$$s = (-1)^q.$$

Let d be the greatest common divisor of the numbers N and q. Observe that d is an odd number and that $d = 1$ in the case of q even.

Setting $k = hN/d$ $(h = 1, 2,..., d)$ in (3.17) we obtain

$$\beta_{hN/d} = (-1)^{hq} \sum_{i=1}^{2^{r-1}} (-1)^{i((h+1)q+1)} \sum_{j=1}^{q} (-1)^j \sin \frac{jh\pi}{d}.$$

Since

$$\begin{aligned}\sum_{i=1}^{2^{r-1}} (-1)^{i((h+1)q+1)} &= 0 \qquad && (h \text{ odd or } q \text{ even}) \\ &= 2^{r-1} && (h \text{ even and } q \text{ odd}),\end{aligned}$$

and, in view of (3.19),

$$\sum_{j=1}^{q} (-1)^j \sin \frac{jh\pi}{d} = \frac{(-1)^q \sin(h(2q+1)\,\pi/2d) - \sin(h\pi/2d)}{2 \cos(h\pi/2d)},$$

we get

$$\begin{aligned} \beta_{(2m+1)N/d} &= 0 \qquad \left(m = 0, 1,\dots, \frac{d-1}{2}\right), \\ \beta_{2mN/d} &= -2^{r-1} \tan \frac{m\pi}{d} \qquad \left(m = 1, 2,\dots, \frac{d-1}{2}\right). \end{aligned} \tag{3.20a}$$

Let $k \neq hN/d$ $(h = 1, 2,\dots, d)$. Making use of the identities

$$\sum_{i=0}^{m} t^i \sin(iu + v) = \frac{\begin{Bmatrix}\sin v + t \sin(u - v) \\ \quad - t^{m+1} \sin((m + 1) u + v) + t^{m+2} \sin(mu + v)\end{Bmatrix}}{1 - 2t \cos u + t^2},$$

$$\begin{aligned} \sum_{i=0}^{m}{}'' t^i \sin(iu + v) = \Bigg\{&\frac{1 - t^2}{2} \sin v + \frac{t}{2} (\sin(u - v) + \sin(u + v)) \\ &+ \left(t^{m+2} - \frac{1 + t^2}{2}\right) \sin(mu + v) \\ &+ t\left(\frac{1}{2} - t^m\right) \sin((m + 1) u + v) \\ &+ \frac{t}{2} \sin((m - 1) u + v)\Bigg\} \Big/ (1 - 2t \cos u + t^2), \end{aligned}$$

we transform in turn the right-hand side of (3.17) to the form

$$\begin{aligned} &\sum_{i=1}^{2^{r-1}} (-1)^{i(q+1)} \frac{(-1)^q \sin \dfrac{(2iq + 1) k\pi}{2N} - \sin \dfrac{(2(i - 1) q + 1) k\pi}{2N}}{2 \cos \dfrac{k\pi}{2N}} \\ &= \frac{(-1)^q}{\cos \dfrac{k\pi}{2N}} \sum_{i=0}^{2^{r-1}}{}'' (-1)^{i(q+1)} \sin \frac{(2iq + 1) k\pi}{2N} \\ &= \frac{(-1)^q}{\cos \dfrac{k\pi}{2N}} \cdot \frac{(-1)^q \sin \dfrac{kq\pi}{N} \left(\cos \dfrac{k(2^r q + 1) \pi}{2N} - \cos \dfrac{k\pi}{2N}\right)}{2\left(1 + (-1)^q \cos \dfrac{kq\pi}{N}\right)}. \end{aligned}$$

Here the symbol $\sum''$ denotes the sum with the first and the last terms halved. Thus we obtain the formula

$$\beta_k = \frac{1}{2} \tan^s \frac{kq\pi}{2N} \left(\cos \frac{k(2^r q + 1) \pi}{2N} - \cos \frac{k\pi}{2N}\right) \Big/ \cos \frac{k\pi}{2N}. \tag{3.21}$$

Formula (3.12) results from substituting (3.18)–(3.21) into (3.14). ∎

Values of $\rho_n^{(r)}$ for $r = 0, 1, \ldots, 5$ and for various n were computed via formulas (3.8), (3.10)–(3.12) and are listed in the Table I. The last column contains values of σ_n defined by (1.6).

TABLE I

n	$\rho_n^{(0)}$	$\rho_n^{(1)}$	$\rho_n^{(2)}$	$\rho_n^{(3)}$	$\rho_n^{(4)}$	$\rho_n^{(5)}$	σ_n
1	1.436	1.273	1.144	1.074	1.037	1.019	1.000
2	1.642	1.436	1.273	1.144	1.074	1.037	1.028
3	1.778	1.552	1.357	1.126	1.065	1.035	1.069
4	1.880	1.642	1.436	1.273	1.144	1.074	1.104
5	1.961	1.716	1.495	1.316	1.122	1.058	1.135
6	2.029	1.778	1.552	1.357	1.126	1.065	1.162
7	2.087	1.832	1.598	1.348	1.153	1.087	1.186
8	2.138	1.880	1.642	1.436	1.273	1.144	1.208
9	2.183	1.923	1.680	1.466	1.295	1.106	1.227
10	2.223	1.961	1.716	1.495	1.316	1.122	1.245
11	2.260	1.997	1.747	1.495	1.308	1.129	1.261
12	2.294	2.029	1.778	1.552	1.357	1.126	1.276
13	2.325	2.059	1.806	1.575	1.325	1.149	1.291
14	2.354	2.087	1.832	1.598	1.348	1.153	1.304
15	2.381	2.113	1.856	1.601	1.352	1.165	1.316
16	2.406	2.138	1.880	1.642	1.436	1.273	1.328
17	2.430	2.161	1.902	1.661	1.451	1.284	1.339
18	2.453	2.183	1.923	1.680	1.466	1.295	1.349
19	2.474	2.204	1.942	1.685	1.464	1.289	1.359
20	2.494	2.223	1.961	1.716	1.495	1.316	1.369
32	2.681	2.406	2.138	1.880	1.642	1.436	1.458
48	2.843	2.567	2.294	2.029	1.778	1.552	1.536
64	2.959	2.681	2.406	2.138	1.880	1.642	1.593
80	3.049	2.770	2.494	2.223	1.961	1.716	1.637
256	3.518	3.238	2.959	2.681	2.406	2.138	1.870

Let us define

$$\gamma_{nrl} = \int_0^\pi \cos(n+l)\,u \operatorname{sgn} D_n^{(r)}(u)\,du$$

$$(n, r = 0, 1,\ldots;\ l = 1, 2,\ldots, 2^r - 1). \tag{3.22}$$

LEMMA 3.3. *For $n = 2^{r-1}m$ $(m, r = 1, 2,\ldots)$ we have*

$$\gamma_{nrl} = 0 \qquad (l = 1, 2,\ldots, 2^r - 1). \tag{3.23}$$

Proof. Let us denote the integrand from the right-hand side of (3.22) by $H_l(u)$ (n, r fixed), i.e.,

$$H_l(u) = \cos(n+l)\,u \operatorname{sgn} D_n^{(r)}(u) \qquad (l = 1, 2,\ldots, 2^r - 1).$$

It can be deduced from (3.4) that

$$\operatorname{sgn} D_n^{(r)}(u) = \operatorname{sgn}(\sin 2^{r-1}u \cdot \sin 2^{r-1}(m+1)\, u).$$

Let l be any number from the set $1, 2,..., 2^r - 1$. Representing l in the form

$$1 = 2^s(2t+1),$$

where $0 \leqslant s \leqslant r-1$, $0 \leqslant t \leqslant 2^{r-s-1} - 1$, and using the fact that the function H_l has a period equal to $\pi/2^{s-1}$, we obtain the equation

$$\gamma_{nrl} = 2^{s-1} \int_0^{\pi/2^{s-1}} H_l(u)\, du.$$

In view of the equality

$$H_l(\pi/2^s - u) = -H_l(\pi/2^s + u) \qquad (0 \leqslant u \leqslant \pi/2^s),$$

the above integral vanishes. ∎

The last two lemmas show the connection of the functions $D_n^{(r)}$ with projection operators discussed in preceding sections.

LEMMA 3.4. *Let* $P \in \mathscr{T}_{np}$ *be defined by* (1.8) *for* $n = 2^{r-1}m$, $p = 2^r - 1$ $(m, r = 1, 2,...)$, *and for* $q_1, q_2,..., q_p \in \Pi_n$ *such that*

$$q_l(1) = 1 - 2^{-r}l \qquad (l = 1, 2,..., p). \tag{3.24}$$

Then we have

$$F_P(1, \cos u) = D_n^{(r)}(u), \tag{3.25}$$

$$\Lambda_P(1) = \rho_m^{(1)}, \tag{3.26}$$

the notation being that of (2.1), (2.2), (3.1), *and* (3.5).

Proof. In accordance with (2.2) we have the formula

$$F_P(1, y) = {\sum_{k=0}^{n}}' T_k(y) + \sum_{l=1}^{p} (1 - l2^{-r})\, T_{n+l}(y),$$

where we used (3.24). Equation (3.25) follows from this formula by substituting $y = \cos u$, and comparing the resulting expression on the right-hand side with definition (3.1).

Equation (3.26) can be easily derived from (2.1), (3.25), (3.5), and (3.6):

$$\Lambda_P(1) = \frac{2}{\pi}\int_{-1}^{1} |F_P(1, y)|\,(1-y^2)^{-1/2}\,dy = \frac{2}{\pi}\int_0^{\pi} |D_n^{(r)}(u)|\,du$$

$$= \rho_n^{(r)} = \rho_m^{(1)}. \quad \blacksquare$$

LEMMA 3.5. *Let* $P \in \hat{\mathscr{T}}_{np}$ *be a symmetric projection defined by* (1.8) *for* $n = 2^{r-1}m + \nu$, $p = 2^r - 2 + \delta - \nu$ ($m = 1, 2,...; r = 2, 3,...; \delta, \nu = 0, 1$), *and for* $q_1, q_1, ..., q_p \in \Pi_n$, *satisfying* (2.7) *and such that*

$$q_{2l-\nu}(0) = (-1)^{l+[n/2]}(1 - 2^{1-r}l) \qquad \{l = 1, 2,..., [(p+\nu)/2]\}. \tag{3.27}$$

Then we have

$$F_P\left(0, \sin\frac{u}{2}\right) = D_{[n/2]}^{(r-1)}(u), \tag{3.28}$$

$$\Lambda_P(0) = \rho_m^{(1)}. \tag{3.29}$$

Proof. It follows from (2.7) that

$$q_{2l+\nu-1}(0) = 0 \qquad \{l = 1, 2,..., [(p+\nu)/2]\}.$$

Formula (2.2) implies the equation

$$F_P(0, y) = \sum_{k=0}^{[n/2]}{}' (-1)^k\, T_{2k}(y) + \sum_{l=1}^{[(p+\nu)/2]} q_{2l-\nu}(0)\, T_{n+2l-\nu}(y).$$

Substitution of $y = \sin(u/2)\{= \cos((\pi - u)/2)\}$ yields the equality

$$F_P\left(0, \sin\frac{u}{2}\right) = \sum_{k=0}^{[n/2]}{}' \cos ku + \sum_{l=1}^{[(p+\nu)/2]} (1 - 2^{r-1}l)\cos([n/2] + l)\,u,$$

where we used assumption (3.27). The right-hand side of the above formula is $D_{[n/2]}^{(r-1)}(u)$ (see (3.1)). Relation (3.28) is proved.

Formula (3.29) follows easily from (2.1), (3.28), (3.5), and (3.6):

$$\Lambda_P(0) = \frac{2}{\pi}\int_{-1}^{1} |F_P(0, y)|\,(1-y^2)^{-1/2}\,dy = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_{[n/2]}^{(r-1)}(u)|\,du$$

$$= \rho_{[n/2]}^{(r-1)} = \rho_m^{(1)}. \quad \blacksquare$$

4. Theorems

Now, we are able to prove the following.

Theorem 4.1. *Let $r = 1, 2,\ldots$ and let M_r denote the smallest natural number such that the inequality*

$$\rho_{M_r}^{(1)} \geqslant \sigma_{N_r} \tag{4.1}$$

holds, where $N_r = 2^{r-1}M_r$, and the notation used is that of (3.11) *and* (1.5). *Let $P \in \mathscr{T}_{np}$ be an operator defined by* (1.8) *for $n = 2^{r-1}m$, $p = 2^r - 1$ ($m = M_r, M_r + 1,\ldots$; $r = 1, 2,\ldots$), and for $q_1, q_2,\ldots, q_p \in \Pi_n$, satisfying* (3.24). *If $1 \in \operatorname{crit}(\Lambda_p)$ then P is a minimal projection from $\mathscr{T}_{np}$ and has the norm*

$$\| P \| = \rho_m^{(1)}. \tag{4.2}$$

Proof. The sequences σ_n and $\rho_n^{(1)}$ are monotonically increasing. Comparing the asymptotic formula

$$\rho_m^{(1)} = \frac{4}{\pi^2} \log m + O(1) \tag{4.3}$$

(see [1]) and

$$\sigma_{2^{r-1}m} = \frac{2}{\pi^2} \log m + \frac{2r-2}{\pi^2} \log 2 + O(1) \tag{4.4}$$

(cf. (1.7)) we see that for r fixed and for m sufficiently large we have

$$\rho_m^{(1)} \geqslant \sigma_{2^{r-1}m}. \tag{4.5}$$

Thus, for $r = 1, 2,\ldots$ there exists the number M_r defined above.

We show that for an operator $P \in \mathscr{T}_{np}$ satisfying the assumptions of the theorem the equation

$$h_l(1) = 0 \qquad (l = 1, 2,\ldots, p) \tag{4.6}$$

holds, where h_l is the function defined by (2.6). By virtue of Theorem 2.1 it follows then that P is a minimal projection from $\mathscr{T}_{np}$.

In order to prove relation (4.6) observe that we have, in view of (2.6), (3.25), and (3.22),

$$\begin{aligned} h_l(1) &= \int_{-1}^{1} T_{n+l}(y) \operatorname{sgn} F_P(1, y)(1 - y^2)^{-1/2}\, dy \\ &= \int_0^{\pi} \cos(n + l)\, u \operatorname{sgn} D_n^{(r)}(u)\, du = \gamma_{nrl} \qquad (l = 1, 2,\ldots, p). \end{aligned}$$

Hence, by virtue of Lemma 3.3, Eq. (4.6) follows.

Formula (4.2) follows from (2.3), (2.4), and (3.26). ∎

Remark 1. The inspection of Table I given in Section 3 shows that

$$M_1 = M_2 = M_3 = M_4 = 1,$$
$$M_5 = 3.$$

It seems that the sequence M_r increases rather fast. For instance we have $M_6 = 8$.

Remark 2. It is clear that for $m < M_r$ the thesis of the theorem fails, as in this case we have $\| P \| = \rho_m^{(1)} < \sigma_n$, which is, in view of (1.4), impossible.

Remark 3. In [3] we have considered the projection $C_n \in \mathscr{T}_{n1}$ given by

$$C_n = S_n + (1/2n)\, a_{n+1}[\cdot]\, U_{n-1}, \tag{4.7}$$

where $U_{n-1}(x) = \sin nu/\sin u$ $(x = \cos u)$. Here $q_1(1) = (1/2n)\, U_{n-1}(1) = \frac{1}{2}$, which means that Eq. (3.24) is satisfied.

We have established numerically that $1 \in \operatorname{crit}(\Lambda_{C_n})$ for $n = 1, 2, 3$. The calculations carried out by Phillips *et al.* [6] confirm this result.

Using the above theorem we conclude that C_n is a minimal operator from $\mathscr{T}_{n1}$ for $n = 1, 2, 3$.

Obviously, we have

COROLLARY 4.1. *For* $n = 2^{r-1}m$ *and* $p = 2^r - 1$ $(m = M_r, M_{r+1}, \ldots;$ $r = 1, 2, \ldots)$ *we have*

$$\inf_{P \in \mathscr{T}_{np}} \Lambda_P(1) = \rho_m^{(1)}.$$

The next theorem of this section is related to symmetric projections.

THEOREM 4.2. *Let*

$$\begin{aligned} M_{\nu r} &= M_r \qquad (\nu = 0) \\ &= M_r' \qquad (\nu = 1) \end{aligned} \qquad (r = 2, 3, \ldots,) \tag{4.8}$$

where M_r *is defined as in Theorem* 4.1, *and* M_r' *is the smalllest natural number such that*

$$\rho_{M_r'}^{(1)} \geqslant \sigma_{N_r'} \qquad (N_r' = 2^{r-1}M_r' + 1).$$

Let $P \in \hat{\mathscr{T}}_{np}$ *be a symmetric projection given by* (1.8) *for* $n = 2^{r-1}m + \nu$, $p = 2^r + \delta - \nu - 2$ $(m = M_{\nu r}, M_{\nu r} + 1, \ldots;\ r = 2, 3, \ldots;\ \delta, \nu = 0, 1)$, *and for* $q_1, q_2, \ldots, q_p \in \Pi_n$, *satisfying conditions* (2.7) *and* (3.27). *If* $0 \in \operatorname{crit}(\Lambda_P)$ *then P is a minimal projection from* $\hat{\mathscr{T}}_{np}$ *and has the norm given by formula* (4.2).

Proof. The asymptotic forms (4.3) and (4.4) imply that for m sufficiently large we have

$$\rho_m^{(1)} \geqslant \sigma_{2^{r-1}m+1}$$

(r being fixed). This means that M_r', defined in the theorem, actually exists for any $r = 2, 3, \ldots$.

Using definition (2.6) and Lemmas 3.5 (Eq. (3.28)) and 3.3 we derive

$$h_{2l-\nu}(0) = \int_{-1}^{1} T_{n+2l+\nu}(y) \operatorname{sgn} F_P(0, y)(1 - y^2)^{-1/2}\, dy$$

$$= \int_0^{\pi} \cos([n/2] + l)\, u \operatorname{sgn} D_{[n/2]}^{(r-1)}(u)\, du = \gamma_{[n/2], r-1, l} = 0$$

$$(l = 1, 2, \ldots, [(p + \nu)/2]).$$

Let $w_1, w_2, \ldots, w_p \in \Pi_n$ satisfy (2.7). Then we have

$$\sum_{i=1}^{p} w_i(0)\, h_i(0) = \sum_{l=1}^{[(p+\nu)/2]} w_{2l-\nu}(0)\, h_{2l-\nu}(0) = 0,$$

and, by virtue of Theorem 2.2, P is a minimal projection from $\hat{\mathscr{T}}_{np}$.

In view of (2.3), (2.4), and Lemma 3.5 (formula (3.29)) we have

$$\| P \| = \Lambda_P(0) = \rho_m^{(1)}. \quad ∎$$

Remark 4. The calculations performed show that

$$M_r' = M_r \qquad (r = 1, 2, \ldots, 6).$$

COROLLARY 4.2. *For* $n = 2^{r-1}m + \nu$ *and* $p = 2^r + \delta - \nu - 2$ $(m = M_{\nu r}, M_{\nu r} + 1, \ldots; r = 2, 3, \ldots; \delta, \nu = 0, 1)$ *we have*

$$\inf_{Q \in \hat{\mathscr{T}}_{np}} \Lambda_Q(0) = \rho_m^{(1)}.$$

As a simple consequence of Corollaries 4.1 and 4.2 we obtain the following.

THEOREM 4.3. *If* $P \in \mathscr{T}_{np}$, *for either* $n = 1, 2, \ldots$ *and* $p = 1$ *or* $n = 2^{r-1}m + \nu$ *and* $p = 2^r + \delta - \nu - 2$ $(m = M_{\nu r}, M_{\nu r} + 1, \ldots; r = 2, 3, \ldots; \delta, \nu = 0, 1)$, *then* P *satisfies the inequality*

$$\| P \| \geqslant \rho_n^{(1)} \qquad (p = 1),$$

$$\geqslant \rho_m^{(1)} \qquad (p > 1).$$

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